The Vasicek Interest Rate Process
Part VI - Interest Rate Model Calibration

Gary Schurman, MBE, CFA

January, 2021

The Vasicek interest rate model is a mathematical model that describes the evolution of the short rate of interest over time. The short rate is the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. Vasicek models the short rate as a Ornstein-Uhlenbeck process. Vasicek's stochastic differential equation that describes the evolution of the short rate r_t in continuous-time is...

$$\delta r_t = \lambda \left(r_\infty - r_t \right) \delta t + \sigma \, \delta W_t \tag{1}$$

When the short rate moves below its long-term mean r_∞ the short rate drift becomes positive and the short rate is pulled upward. When the short rate moves above its long-term mean the short rate drift becomes negative and the short rate is pulled downward. The speed at which the drift is pulled upward of downward is given by the positive valued parameter λ, which measures the speed of mean reversion. The greater the speed the faster the process reverts toward the long-term mean. Random shocks are introduced via the variables σ, which is the annualized short rate volatility, and δW_t, which is the change in the driving Brownian motion over the infinitesimally short time interval $[t, t + \delta t]$.

Using Equation (1) above the equation for the evolution of the short rate of interest in discrete time is...

$$\Delta \, r_t = \lambda \left(r_\infty - r_t \right) \Delta_t + \epsilon \; \text{...where...} \; \Delta_t > \delta t \; \text{...and...} \; \epsilon = \text{error term} \tag{2}$$

We will use discrete-time Equation (2) above to calibrate our model to actual market interest rate data.

Our Hypothetical Problem

We are told to use the Federal Funds Rate as a proxy for the risk-free short-rate. The federal funds rate is the target interest rate set by the Fed at which commercial banks borrow and lend their excess reserves to each other overnight. We are given the following table of monthly Federal Funds Rates over the time interval $[07 - 1954, 12 - 2021]$.

Table 1: Federal Funds Rate (FRED time series FEDFUNDS)

Month	Year	Rate
7	1954	0.80%
8	1954	1.22%
9	1954	1.07%
...	...	...
11	2021	0.08%
12	2021	0.08%

Question 1: Graph the Federal Funds Rate time series.

Question 2: Estimate the parameters for the short rate model.

Question 3: What is the goodness of fit of our short rate model?

Modeling The Short Rate In Discrete-Time

The equation for the random short rate at time $t + \Delta t$ as a function of the known short rate at time t is...

$$r_{t+\Delta t} = r_t + \Delta \, r_t \tag{3}$$

Using Equation (2) above we can rewrite Equation (3) above as...

$$\begin{aligned} r_{t+\Delta t} &= r_t + \lambda\,(r_\infty - r_t)\,\Delta_t + \epsilon_t \\ &= r_t + \lambda\,r_\infty\,\Delta_t - \lambda\,r_t\,\Delta_t + \epsilon_t \\ &= r_t\,(1 - \lambda\,\Delta t) + \lambda\,r_\infty\,\Delta_t + \epsilon_t \end{aligned} \tag{4}$$

We will define the variable z_t to be an independent random variate drawn at time t from a normal distribution with mean zero and variance one. We will make the following definitions...

$$a = 1 - \lambda\,\Delta t \text{ ...and... } b = \lambda\,r_\infty\,\Delta_t \text{ ...and... } \epsilon_t = \sigma\sqrt{\Delta t}\,z_t \tag{5}$$

Using the definitions in Equation (5) above we can rewrite Equation (4) above as...

$$r_{t+\Delta t} = a\,r_t + b + \epsilon_t \tag{6}$$

Isolating the error term we can rewrite Equation (6) above as...

$$\epsilon_t = r_{t+\Delta t} - (a\,r_t + b) \tag{7}$$

Model Calibration Using A Least Squares Methodology

We will start by formatting the data in Table 1 above where r_i is the short rate at time index i...

Table 2: Model Data

Time Period	Index i	A $r(i)$	B $r(i+1)$	C $r(i)^2$	D $r(i)r(i+1)$	E B-A	F (B-A)(B-A)
07-1954	1	0.0080	0.0122	0.0001	0.0001	0.0042	0.00001764
08-1954	2	0.0122	0.0107	0.0001	0.0001	-0.0015	0.00000225
09-1954	3	0.0107	0.0085	0.0001	0.0001	-0.0022	0.00000484
—	—	—	—	—	—	—	—
10-2021	808	0.0008	0.0008	0.0000	0.0000	0.0000	0.00000000
11-2021	809	0.0008	0.0008	0.0000	0.0000	0.0000	0.00000000
Total	—	37.5722	37.5650	2.8049	2.7951	-0.0720	0.01955702

The equation for the change in time in years is...

$$\Delta t = \frac{1}{\text{number of data periods in one year}} \tag{8}$$

We will also rewrite Equations (6) and (7) above to be consistent with the data series in Table 2 above

$$r_{i+1} = a\,r_i + b + \epsilon_i \text{ ...such that... } \epsilon_i = r_{i+1} - (a\,r_i + b) \tag{9}$$

We will define the term SSE to be the sum of squared errors. Using Equation (9) above and the Table 2 above the equation for the sum of squared errors is...

$$SSE = \sum_{i=1}^{N} \epsilon_i^2 = \sum_{i=1}^{N} \Big(r_{i+1} - (a\,r_i + b)\Big)^2 \text{ ...where... } N = \text{Number of data points} \tag{10}$$

Per Appendix Equations (34) and (35) below the derivatives of Equation (10) above are...

$$\frac{\delta\,SSE}{\delta\,a} = 2\left[\sum_{i=1}^{N} a\,r_i^2 + \sum_{i=1}^{N} b\,r_i - \sum_{i=1}^{N} r_{i+1}\,r_i\right] \tag{11}$$

$$\frac{\delta\,SSE}{\delta\,b} = 2\left[N\,b + \sum_{i=1}^{N} a\,r_i - \sum_{i=1}^{N} r_{i+1}\right] \tag{12}$$

Our goal is to set the values of model parameters a and b such that the sum of squared errors (Equation (10) above) is minimized. To do this we set derivative Equations (11) and (12) above equal to zero and then solve for parameters a and b. Using Appendix Equations (36) and (38) below the equations for model parameters a and b are...

$$a = \left[\sum_{i=1}^{N} r_{i+1}\, r_i - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_{i+1}\right] \Bigg/ \left[\sum_{i=1}^{N} r_i^2 - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_i\right] \tag{13}$$

$$b = \frac{1}{N}\left[\sum_{i=1}^{N} r_{i+1} - \sum_{i=1}^{N} a\, r_i\right] \tag{14}$$

Once we have values for model parameters a and b we use Equation (5) above to solve for the mean reversion parameter lambda...

$$\text{if... } a = 1 - \lambda\,\Delta t \text{ ...then... } \lambda = \frac{1-a}{\Delta t} \tag{15}$$

Given Equation (15) above we can then use Equation (5) above to solve for the long-term short rate...

$$\text{if... } b = \lambda\, r_\infty\, \Delta_t = \frac{1-a}{\Delta t}\, r_\infty\, \Delta_t = (1-a)\, r_\infty \text{ ...then... } r_\infty = \frac{b}{1-a} \tag{16}$$

The equation for the variance of the error term in Equations (6) and (7) above is...

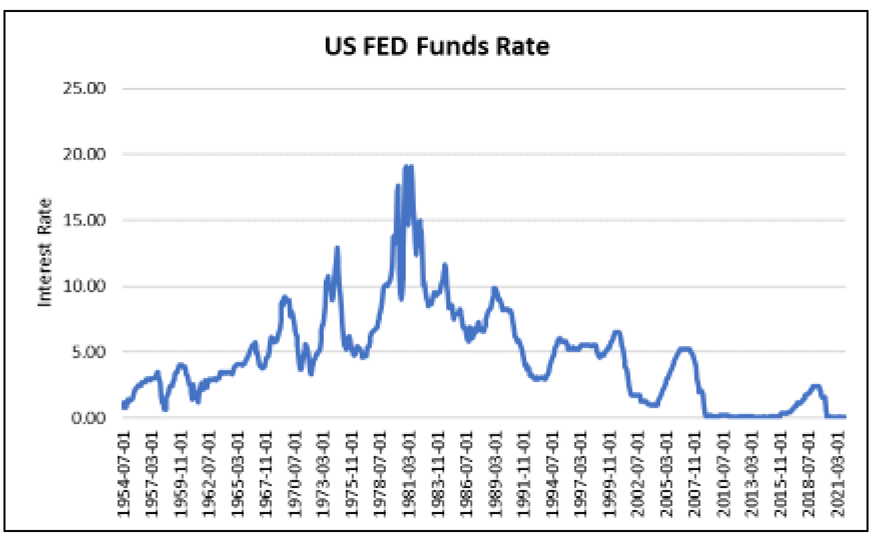
$$Var(\epsilon) = \sigma^2 \Delta_t = \mathbb{E}\Big[\epsilon^2\Big] - \Big[\mathbb{E}\Big[\epsilon\Big]\Big]^2 = \frac{1}{N}\sum_{i=1}^{N} \epsilon_i^2 - \left[\frac{1}{N}\sum_{i=1}^{N} \epsilon_i^2\right]^2 \tag{17}$$

Using Equation (17) above and solving for the model parameter sigma we get...

$$\sigma = \sqrt{\frac{Var(\epsilon)}{\Delta_t}} = \sqrt{\frac{1}{\Delta_t}\left[\frac{1}{N}\sum_{i=1}^{N} \epsilon_i^2 - \left[\frac{1}{N}\sum_{i=1}^{N} \epsilon_i\right]^2\right]} \tag{18}$$

The Answers To Our Hypothetical Problem

Question 1: Graph the Federal Funds Rate time series.



Question 2: Estimate the parameters for the short rate model.

Using Equation (8) above the equation for the change in time is...

$$\Delta t = \frac{1}{\text{number of data periods in one year}} = \frac{1}{12} = 0.08333 \tag{19}$$

Using the data in Table 2 above the value of N is...

$$N = \text{Number of data points} = 809 \tag{20}$$

Using Equation (13) above and the data in Table 2 above the equation for model parameter a is...

$$a = \left[2.7951 - \frac{1}{12} \times 37.5722 \times 37.5650\right] \bigg/ \left[2.8049 - \frac{1}{809} \times 37.5722 \times 37.5722\right] = 0.99106 \tag{21}$$

Using Equations (14) and (21) above and the data in Table 2 above the equation for model parameter b is...

$$b = \frac{1}{809} \times \left[37.5650 - 0.99106 \times 37.5722\right] = 0.000406 \tag{22}$$

Using Equations (15), (19) and (21) above the equation for mean reversion parameter lambda is...

$$\lambda = \frac{1-a}{\Delta t} = \frac{a - 0.99106}{0.08333} = 0.10728 \tag{23}$$

Using Equations (16), (21) and (22) above the equation for the long-term short rate is...

$$r_{\infty} = \frac{b}{1-a} = \frac{0.000406}{1 - 0.99106} = 0.04545 \tag{24}$$

Using Equations (18), (19), (20) above and the data in Table 2 above the equation for model parameter sigma is...

$$\sigma = \sqrt{\frac{1}{\Delta_t}\left[\frac{1}{N}\sum_{i=1}^{N}\epsilon_i^2 - \left[\frac{1}{N}\sum_{i=1}^{N}\epsilon_i\right]^2\right]} = \sqrt{\frac{1}{0.08333} \times \left[\frac{1}{809} \times 0.01956 - \left[\frac{1}{809} \times -0.0072\right]\right]} = 0.01703 \tag{25}$$

Using Equation (4) above and the parameter estimates above our short rate model becomes...

$$r_{t+0.08333} = r_t \times (1 - 0.10728 \times 0.08333) + 0.10728 \times 0.04545 \times 0.08333 + 0.01703 \times \sqrt{0.08333} \times z_t \tag{26}$$

Question 3: What is the goodness of fit of our short rate model?

Regression model: Using Equation (26) above the equation for the expected short rate at time $t + \Delta t$ as a function of the known short rate at time t is...

$$\mathbb{E}\left[r_{t+0.08333}\right] = r_t \times (1 - 0.10728 \times 0.08333) + 0.10728 \times 0.04545 \times 0.08333 \text{ ...given that... } \mathbb{E}\left[z_t\right] = 0 \tag{27}$$

Mean model: Using the data in Table 2 above our mean model for the short rate at time $t + \Delta t$ is...

$$\mathbb{E}\left[r_{t+0.08333}\right] = \frac{1}{N}\sum_{i=1}^{N} Y_i = \frac{1}{809} \times 37.5722 = 0.0465 \tag{28}$$

We will define the variable SST to be the sum of squared errors using the mean model in Equation (28) above. Using the data in Table 2 above the sum of squared errors using the mean model is...

$$SST = 1.06053 \tag{29}$$

Using Equation (29) above and the data in Table 2 above the equation for goodness of fit is... [2]

$$\text{R-squared} = \frac{SST - SSE}{SST} = \frac{1.06053 - 0.01956}{1.06053} = 0.9816 \tag{30}$$

Appendix

A. We want to expand the following equation...

$$\begin{aligned} SSE &= \sum_{i=1}^{N}\Big(r_{i+1} - (a\,r_i + b)\Big)^2 \\ &= \sum_{i=1}^{N}\Big(r_{i+1}^2 - 2\,r_{i+1}\,(a\,r_i + b) + (a\,r_i + b)^2\Big) \\ &= \sum_{i=1}^{N} r_{i+1}^2 - 2\sum_{i=1}^{N} a\,r_{i+1}\,r_i - 2\sum_{i=1}^{N} b\,r_{i+1} + \sum_{i=1}^{N}(a\,r_i + b)^2 \end{aligned} \tag{31}$$

Note the following equation...

$$(a\, r_i + b)^2 = a^2 r_i^2 + b^2 + 2\, a\, b\, r_i \tag{32}$$

Using Equation (32) above we can rewrite Equation (31) above as...

$$SSE = \sum_{i=1}^{N} r_{i+1}^2 - 2 \sum_{i=1}^{N} a\, r_{i+1}\, r_i - 2 \sum_{i=1}^{N} b\, r_{i+1} + \sum_{i=1}^{N} a^2 r_i^2 + \sum_{i=1}^{N} b^2 + 2 \sum_{i=1}^{N} a\, b\, r_i \tag{33}$$

B. The derivative of Equation (33) above with respect to the parameter a is...

$$\frac{\delta\, SSE}{\delta\, a} = -2 \sum_{i=1}^{N} r_{i+1}\, r_i + 2 \sum_{i=1}^{N} a\, r_i^2 + 2 \sum_{i=1}^{N} b\, r_i = 2 \Bigg[\sum_{i=1}^{N} a\, r_i^2 + \sum_{i=1}^{N} b\, r_i - \sum_{i=1}^{N} r_{i+1}\, r_i \Bigg] \tag{34}$$

C. The derivative of Equation (33) above with respect to the parameter b is...

$$\frac{\delta\, SSE}{\delta\, b} = -2 \sum_{i=1}^{N} r_{i+1} + 2 \sum_{i=1}^{N} b + 2 \sum_{i=1}^{N} a\, r_i = 2 \left[N\, b + \sum_{i=1}^{N} a\, r_i - \sum_{i=1}^{N} r_{i+1} \right] \tag{35}$$

D. The Least Squares estimate for the variable b in the short rate Equation (6) is...

$$\frac{\delta\, SSE}{\delta\, b} = 0$$

$$2 \left[N\, b + \sum_{i=1}^{N} a\, r_i - \sum_{i=1}^{N} r_{i+1} \right] = 0$$

$$N\, b + \sum_{i=1}^{N} a\, r_i - \sum_{i=1}^{N} r_{i+1} = 0$$

$$\frac{1}{N} \left[\sum_{i=1}^{N} r_{i+1} - a \sum_{i=1}^{N} r_i \right] = b \tag{36}$$

E. The Least Squares estimate for the variable a in the short rate Equation (6) is...

$$\frac{\delta\, SSE}{\delta\, a} = 0$$

$$2 \Bigg[\sum_{i=1}^{N} a\, r_i^2 + \sum_{i=1}^{N} b\, r_i - \sum_{i=1}^{N} r_{i+1}\, r_i \Bigg] = 0$$

$$\sum_{i=1}^{N} a\, r_i^2 + \sum_{i=1}^{N} b\, r_i - \sum_{i=1}^{N} r_{i+1}\, r_i = 0 \tag{37}$$

Substituting Equation (36) above for the variable b in Equation (37) above we get...

$$\sum_{i=1}^{N} a\, r_i^2 + \sum_{i=1}^{N} \frac{1}{N} \Bigg[\sum_{i=1}^{N} r_{i+1} - \sum_{i=1}^{N} a\, r_i \Bigg] r_i - \sum_{i=1}^{N} r_{i+1}\, r_i = 0$$

$$\sum_{i=1}^{N} a\, r_i^2 + \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_{i+1} - \frac{1}{N}\, a \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_i - \sum_{i=1}^{N} r_{i+1}\, r_i = 0$$

$$a \sum_{i=1}^{N} r_i^2 - \frac{1}{N}\, a \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_i = \sum_{i=1}^{N} r_{i+1}\, r_i - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_{i+1}$$

$$a \Bigg[\sum_{i=1}^{N} r_i^2 - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_i \Bigg] = \sum_{i=1}^{N} r_{i+1}\, r_i - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_{i+1}$$

$$\Bigg[\sum_{i=1}^{N} r_{i+1}\, r_i - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_{i+1} \Bigg] \Bigg/ \Bigg[\sum_{i=1}^{N} r_i^2 - \frac{1}{N} \sum_{i=1}^{N} r_i \sum_{i=1}^{N} r_i \Bigg] = a \tag{38}$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *Univariate Ordinary Least Squares Estimator,* May, 2011.